ISSN 0976 – 1411

Available online at www.internationaleJournals.com

International eJournals

INTERNATIONAL eJOURNAL OF MATHEMATICS AND ENGINEERING

International eJournal of Mathematics and Engineering 212 (2013) 2084 - 2093

www.InternationaleJournals.com

SOLUTION OF TELEGRAPH EQUATIONS BY DIFFERENTIAL TRANSFORM METHOD

S.Vishwa Prasad Rao[1], P. S. Rama Chandra Rao[2],

C. Bala Rama Krishna[3] and B. Nageswara Rao[4]

[1]Department of Mathematics, S. R. Engineering College, Ananthasagar, Hasanparthy, Warangal - 506371, Andhra Pradesh, India, email: vishwa72@yahoo.com

[2]Department of Mathematics, Kakatiya Institute of Technology and Science, Warangal - 506015, Andhra Pradesh, India, email: anuradhawgl@yahoo.co.in

[3]Department of Mathematics, Chaitanya Degree & PG College (Autonomous), Warangal-506001, Andhra Pradesh, India, email: cbrk2004@gmail.com

[4]Department of Basic Science and Humanities (Mathematics), Coastal Institute of Technology and Management, Narapam, Kothavalasa, Vijayanagaram, Andhra Pradesh, India email: dr.bnrao@citm.ac.in

Abstract: Differential transform method has been applied to solve many linear and non-linear differential equations. In this article, this method is applied to solve the partial differential equation, called Telegraph equation.

Keywords: Differential Transform Method; Telegraph Equations.

Mathematics Subject Classification 2010: 33E12, 35R11, 35A22.

1. Introduction

Telegraph equations appear in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems. The general linear telegraph equation is

$$u_{tt} + au_t + bu = cu_{xx}$$

with the initial conditions:

$u(x,0) = \alpha$, $u_t(x,0) = \beta$, where α, β are functions of x.

In this article, we apply the Differential transform method (DTM) for solving the telegraph equations.

2. Basic idea of differential transform method

The basic definitions and fundamental operations of the two dimensional differential transform are as follows. Consider a function of two variable w(x, y), be analytic in the domain S and let $(x, y) = (x_0, y_0)$ in this domain. The function w(x, y) is then represented by one series whose centre at located at (x_0, y_0) . The differential transform of the function w(x, y) is the form

$$W(x,y) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)} \quad (2.1)$$

where w(x, y) is the original function and W(k,h) is the transformed function.

The differential inverse transform of W(k,h) is defined as

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)(x-x_0)^k (y-y_0)^h \quad (2.2)$$

In the application when (x_0, y_0) taken as (0,0) and from (2.1) and (2.2) we have,

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right] x^k y^h \quad (2.3)$$

Table

The operations for the two-dimensional differential transform method are listed in the following Table

Original function	Transform function
1) $w(x,y) = u(x,y) \pm v(x,y)$	$W(k,h) = U(k,h) \pm V(k,h)$
2) $w(x,y) = \alpha u(x,y)$	$W(k,h) = \alpha U(k,h)$

3) $w(x,y)=\dfrac{\partial u(x,y)}{\partial x}$ $\quad W(k,h)=(k+1)U(k+1,h)$

4) $w(x,y)=\dfrac{\partial u(x,y)}{\partial y}$ $\quad W(k,h)=(h+1)U(k,h+1)$

5) $w(x,y)=u(x,y)v(x,y)$ $\quad W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$

6) $w(x,y)=x^m y^n$ $\quad W(k,h)=\delta(k-m,h-n);\delta(k-m)=\begin{cases}1, k=m\\0, k\neq m\end{cases}$

$$and\,\delta(h-n)=\begin{cases}1, h=n\\0, h\neq n\end{cases}$$

3 Numerical Examples

In this part, differential transform method will be applied for solving equations of linear and non-linear forms. The results reveal that the method is very effective and simple.

Example 1. Consider the Telegraph equation

$$u_{xx}=u_{tt}+2u_t+u \tag{3.1}$$

with the initial conditions

$$u(x,0)=e^{2x} \text{ and } u_t(x,0)=-e^{x} \tag{3.2}$$

The transformed version of (3.1) is

$$(k+1)(k+2)U(k+2,h)=(h+1)(h+2)U(k,h+2)+2(h+1)U(k,h+1)+U(k,h) \tag{3.3}$$

The transformed version of (3.2) is

$$U(k,0)=\frac{2^k}{k!} \text{ and } U(k,1)=\frac{-1}{k!} \tag{3.4}$$

substituting (3.4) in (3.3),

we get, $U(0,2)=\dfrac{5}{2}$, $\quad U(1,2)=4$, $\quad U(0,3)=\dfrac{-5}{3}$,

$$U(1,3)=\frac{-8}{3},\qquad U(1,4)=\frac{25}{12},\qquad U(2,2)=\frac{7}{2},$$
$$U(2,3)=\frac{-7}{2},\qquad U(0,4)=\frac{4}{3},\qquad U(0,5)=\frac{-4}{5},$$
$$U(3,2)=\frac{13}{6},\qquad U(0,6)=\frac{7}{18},\qquad U(3,3)=\frac{-13}{9},$$
$$U(1,5)=\frac{-43}{20},\qquad U(4,2)=\frac{25}{24},\ldots\ldots\ldots\ldots$$

the solution of equation (3.1) is

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$
$$=\left(1+2x+2x^2+\frac{4}{3}x^3+\ldots\ldots\ldots\right)-t\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots\ldots\ldots\right)$$
$$+t^2\left(\frac{5}{2}+4x+\frac{7}{2}x^2+\frac{13}{6}x^3+\ldots\ldots\ldots\right)-t^3\left(\frac{5}{3}+\frac{8}{3}x+\frac{7}{2}x^2+\frac{13}{9}x^3+\ldots\ldots\ldots\right)+\ldots\ldots\ldots$$
$$u(x,t)=e^{2x}-te^{x}+t^2\left(\frac{5}{2}+4x+\frac{7}{2}x^2+\frac{13}{6}x^3+\ldots\ldots\ldots\right)-$$
$$-t^3\left(\frac{5}{3}+\frac{8}{3}x+\frac{7}{2}x^2+\frac{13}{9}x^3+\ldots\ldots\ldots\right)+\ldots\ldots\ldots$$

Example 2. Consider the Telegraph equation

$$u_{xx}=u_{tt}+4u_t+4u \tag{3.5}$$

with the initial conditions

$$u(x,0)=e^{x}\ \text{and}\ \ u_t(x,0)=-2e^{x} \tag{3.6}$$

The transformed version of (3.5) is

$$(k+1)(k+2)U(k+2,h)=(h+1)(h+2)U(k,h+2)+4(h+1)U(k,h+1)+4U(k,h) \tag{3.7}$$

The transformed version of (3.6) is

$$U(k,0)=\frac{1}{k!}\quad \text{and}\ U(k,1)=\frac{-2}{k!} \tag{3.8}$$

substituting (3.8) in (3.7),

we get, $U(0,2)=\frac{-5}{2}$, $U(1,2)=\frac{5}{2}$, $U(0,3)=\frac{8}{3}$,

$U(1,3)=\frac{-7}{3}$, $U(0,4)=\frac{-13}{8}$, $U(2,2)=\frac{5}{4}$,
$U(3,2)=\frac{7}{6}$, $U(2,3)=\frac{-7}{6}$, $U(1,4)=\frac{25}{12}$,

$U(0,5)=\frac{13}{20}$, $U(4,2)=\frac{5}{48}$, $U(3,3)=\frac{-25}{18}$,

$U(2,4)=\frac{11}{16}$, $U(1,5)=\frac{33}{10}$,..................................

the solution of equation (3.5) is

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$

$$=\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+...........\right)-t\left(2+2x+x^2+\frac{1}{3}x^3+...........\right)$$

$$+t^2\left(\frac{-5}{2}+\frac{5}{2}x+\frac{5}{4}x^2+\frac{7}{6}x^3+...........\right)+t^3\left(\frac{8}{3}-\frac{7}{3}x-\frac{7}{6}x^2-\frac{25}{18}x^3+...........\right)+...............$$

$$u(x,t)=e^x-2te^x+t^2\left(\frac{-5}{2}+\frac{5}{2}x+\frac{5}{4}x^2+\frac{7}{6}x^3+...........\right)+$$
$$+t^3\left(\frac{8}{3}-\frac{7}{3}x-\frac{7}{6}x^2-\frac{25}{18}x^3+...........\right)+..............$$

Example 3. Consider the Telegraph equation

$$u_{xx}=u_{tt}+2u_t+2u \tag{3.9}$$

with the initial conditions

$$u(x,0)=e^x \text{ and } u_t(x,0)=e^{2x} \tag{3.10}$$

The transformed version of (3.9) is

$$(k+1)(k+2)U(k+2,h)=(h+1)(h+2)U(k,h+2)+2(h+1)U(k,h+1)+2U(k,h) \tag{3.11}$$

The transformed version of (3.10) is

$$U(k,0)=\frac{1}{k!} \quad \text{and} \quad U(k,1)=\frac{2^k}{k!} \tag{3.12}$$

substituting (3.12) in (3.11),

we get, $U(0,2)=-1$, $\quad U(1,2)=\frac{-5}{2}$, $\quad U(0,3)=1$,

$$U(2,2)=\frac{-9}{4}, \quad U(1,3)=\frac{7}{3}, \quad U(0,4)=\frac{-17}{24},$$

$$U(3,2)=\frac{-17}{12}, \quad U(2,3)=\frac{13}{6}, \quad U(1,4)=\frac{-11}{9},$$

$$U(0,5)=\frac{1}{3},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

the solution of equation (3.9) is

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$

$$=\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots\ldots\ldots\right)+t\left(1+2x+2x^2+\frac{4}{3}x^3+\ldots\ldots\ldots\right)-$$

$$-t^2\left(1+\frac{5}{2}x+\frac{9}{4}x^2+\frac{17}{12}x^3+\ldots\ldots\ldots\right)+t^3\left(1+\frac{7}{3}x+\frac{13}{6}x^2+\ldots\ldots\ldots\right)+\ldots\ldots\ldots\ldots$$

$$u(x,t)=e^x-2te^x+t^2\left(\frac{-5}{2}+\frac{5}{2}x+\frac{5}{4}x^2+\frac{7}{6}x^3+\ldots\ldots\ldots\right)+$$

$$+t^3\left(\frac{8}{3}-\frac{7}{3}x-\frac{7}{6}x^2-\frac{25}{18}x^3+\ldots\ldots\ldots\right)+\ldots\ldots\ldots\ldots$$

Example 4. Consider the Telegraph equation

$$u_{xx}=u_{tt}+2u_t+u^2-e^{2x-4t}+e^{x-2t} \tag{3.13}$$

with the initial conditions

$$u(x,0)=e^x \quad \text{and} \quad u_t(x,0)=e^{-x} \tag{3.14}$$

The transformed version of (3.13) is

$$(k+1)(k+2)U(k+2,h)$$

$$=(h+1)(h+2)U(k,h+2)+2(h+1)U(k,h+1)$$

$$+\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)U(k-r,s)-\frac{2^k(-4)^h}{k!h!}+\frac{(-2)^h}{k!h!} \qquad (3.15)$$

The transformed version of (3.14) is

$$U(k,0)=\frac{1}{k!} \quad \text{and } U(k,1)=\frac{(-1)^k}{k!} \qquad (3.16)$$

substituting (3.16) in (3.15),

we get, $U(0,2)=-1,\quad U(1,2)=1,\quad U(0,3)=\frac{1}{6},$

$U(2,2)=\frac{-1}{2},\quad U(1,3)=\frac{-11}{6},\quad U(0,4)=\frac{5}{12},$

$U(3,2)=\frac{-1}{6},\quad U(2,3)=\frac{-5}{4},\quad U(1,4)=\frac{13}{6},$

$U(0,5)=\frac{-1}{24},$………………………………

the solution of equation (3.13) is

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$

$$=\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots\ldots\ldots\ldots\right)+t\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\ldots\ldots\ldots\ldots\right)-$$

$$-t^2\left(1+\frac{5}{2}x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots\ldots\ldots\ldots\right)+t^3\left(\frac{1}{6}-\frac{11}{6}x-\frac{5}{4}x^2+\ldots\ldots\ldots\ldots\right)+\ldots\ldots\ldots\ldots\ldots$$

$$u(x,t)=e^x+te^{-x}-t^2\left(1+\frac{5}{2}x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots\ldots\ldots\ldots\right)+$$

$$+t^3\left(\frac{1}{6}-\frac{11}{6}x-\frac{5}{4}x^2+\ldots\ldots\ldots\ldots\right)+\ldots\ldots\ldots\ldots\ldots$$

Example 5. Consider the Telegraph equation

$$u_{xx}=u_{tt}+u_t+u \qquad (3.17)$$

with the initial conditions

$$u(x,0)=\sin x \quad \text{and} \quad u_t(x,0)=0 \qquad (3.18)$$

The transformed version of (3.17) is

$$(k+1)(k+2)U(k+2,h)$$
$$=(h+1)(h+2)U(k,h+2)+(h+1)U(k,h+1)+U(k,h) \tag{3.19}$$

The transformed version of (3.18) is

$$U(k,0)=\begin{cases} 0, k=0,2,4..... \\ \frac{1}{k!}, k=1,5,9...... \\ \frac{-1}{k!}, k=3,7,11...... \end{cases} \quad \text{and } U(k,1)=0 \tag{3.20}$$

substituting (3.20) in (3.19),

we get, $U(1,0)=1$, $U(1,2)=-1$, $U(1,3)=\frac{2}{3}$,

$U(3,0)=\frac{-1}{6}$, $U(3,2)=\frac{1}{6}$, $U(3,3)=\frac{-1}{9}$,

$U(3,4)=\frac{1}{36}$, $U(5,0)=\frac{1}{120}$, $U(5,2)=\frac{-1}{120}$,

$U(5,3)=\frac{1}{180}$, $U(5,4)=\frac{-1}{720}$, $U(7,0)=\frac{-1}{5040}$,

$U(7,2)=\frac{1}{5040}$, $U(7,3)=\frac{-1}{7560}$,.................................

the solution of equation (3.17) is

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$

$$=\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7............\right)-t^2\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7.............\right)-$$

$$+\frac{2}{3}t^3\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7............\right)-\frac{1}{6}t^4\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7.............\right)-$$

$$u(x,t)=\sin x\left(1-t^2+\frac{2}{3}t^3-\frac{1}{6}t^4+......................\right)$$

Example 6. Consider the Telegraph equation

$$16u_{xx} = u_{tt} + 5u_t + 6u \tag{3.21}$$

with the initial conditions

$$u(x,0) = e^x \text{ and } u_t(x,0) = x \tag{3.22}$$

The transformed version of (3.21) is

$$16(k+1)(k+2)U(k+2,h) = (h+1)(h+2)U(k,h+2) + 5(h+1)U(k,h+1) + 6U(k,h) \tag{3.23}$$

The transformed version of (3.22) is

$$U(k,0) = \frac{1}{k!} \quad \text{and } U(k,1) = \begin{cases} 1, k = 1 \\ 0, k = 0, 2, 3, ... \end{cases} \tag{3.24}$$

substituting (3.24) in (3.23),

we get, $U(0,2) = 5$, $U(1,2) = \frac{5}{2}$, $U(0,3) = \frac{-25}{3}$,

$U(2,2) = \frac{15}{4}$, $U(1,3) = \frac{-31}{6}$, $U(0,4) = \frac{215}{12}$,

$U(3,2) = \frac{5}{6}$, $U(2,3) = \frac{-25}{4}$, $U(1,4) = \frac{75}{8}$,

$U(0,5) = \frac{-65}{12}$, ...

the solution of equation (3.21) is

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h) x^k t^h$$

$$= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ...\right) + xt + t^2\left(5 + \frac{5}{2}x + \frac{15}{4}x^2 + \frac{5}{6}x^3 + ...\right)$$

$$-t^3\left(\frac{25}{3} + \frac{31}{6}x + \frac{25}{4}x^2 + ...\right) + ...$$

$$u(x,t) = e^x + xt + t^2\left(5 + \frac{5}{2}x + \frac{15}{4}x^2 + \frac{5}{6}x^3 + ...\right)$$

$$-t^{3}\left(\frac{25}{3}+\frac{31}{6}x+\frac{25}{4}x^{2}+...........\right)+...............$$

4 Conclusions and Discussion

The Differential transform method is a powerful method, which has provided an efficient potential for the solution of physical applications modeled by non-linear differential equations. The main goal of article has been to derive an approximation to the solution of telegraph equation. We have achieved this by applying Differential transform method. In Examples, the approximation can be obtained to any desired number of terms to increase the level of accuracy.

5 References:

[1] C. K Chen, and Ho. S. H. (1999). Solving partial differential equations by two-dimensional differential transform method. Appl Math Comput 106, pp.171–179.

[2] J. Biazar, H. Ebrahimi and Z. Ayati, An approximation to the solution of telegraph equation by variational iteration method, Numerical Methods for Partial Differential Equations 25(4) (2009), pp. 797–801.

[3] P.S. Rama Chandra Rao, Special Multistep Methods Based on Numerical Differentiation for Solving the Initial Value Problem, Applied Mathematics and Computation Elsvier, 181, p.500-510, 2006.

[4] S. Vishwa Prasad Rao and P.S. Rama Chandra Rao, Transform Technique For The Solution Of A Class Of Differential Equation, International Journal Of Applied Mathematical Analysis and Applications.Vol 3,No.2, p.161-166, 2008

[5] S. Vishwa Prasad Rao, P.S. Rama Chandra Rao and C. Prabhakara Rao,TransformTechnique For The Solution Of First Order Linear And Non-Linear Differential Equations, International J. of Math. Sci. & Engg. Appls., Vol. 5 No. III , pp. 49-55, 2011

[6] S. Vishwa Prasad Rao, P.S. Rama Chandra Rao and C. Prabhakara Rao, " Solution Of Differential Equation From The Transform Techniqe" International Journal of Computational Science and Mathematics. Volume 3, Number 1 pp. 121-125 (2011),

[7] Zhou, J.K. (1986). Differential transformation and its applications for electrical circuits. Wuhan, China: Huazhong University Press.