

ON $(\bar{N}, p_n)(E, q)$ PRODUCT SUMMABILITY OF FOURIER SERIES

¹Mahendra Misra, ²U.K.Misra, ³B.P.Padhy and ⁴M.K.Muduli

¹Government Science College

Malkangiri, Orissa

e-mail: mahendramisra@2007.gmail.com

²Department of Mathematics

Berhampur University

Berhampur – 760007, Orissa, India

e-mail: umakanta_misra@yahoo.com

³Roland Institute of Technology

Golanthara-760008, Odisha, India.

e-mail: iraady@gmail.com

⁴Roland Institute of Technology

Golanthara-760008, Odisha, India.

malayamudulimath@gmail.com

Abstract: In this paper, a theorem on $(\bar{N}, p_n)(E, q)$ product summability of Fourier series is proved.

Keywords: (\bar{N}, p_n) mean, $(\bar{N}, p_n)(E, q)$ product mean and Fourier series.

Mathematics subjection classification: 42B05, 42B08.

1. Introduction:

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $p = \{p_n\}$ be a sequence of non-negative, non-increasing real constants such that

$$(1.1) \quad P_n = \sum_{\nu=0}^n p_\nu \rightarrow \infty \text{ as } n \rightarrow \infty, \quad P_{-i} = p_{-i} = 0 \quad \forall i \geq 1$$

The sequence –to–sequence transformation

$$(1.2) \quad t_n = \frac{1}{P_n} \sum_{\nu=0}^n p_\nu s_\nu$$

defines the sequence $\{t_n\}$ of the (\bar{N}, p_n) - mean of the sequence $\{s_n\}$, [1].

Clearly, (\bar{N}, p_n) is regular if

$$(1.3) \quad \begin{cases} (i) P_n \rightarrow \infty, \text{ as } n \rightarrow \infty. \\ (ii) \sum_{i=0}^n p_i \leq C|P_n|, \text{ as } n \rightarrow \infty. \end{cases}$$

If $t_n \rightarrow s$, as $n \rightarrow \infty$, then the series $\sum a_n$ is said to be (\bar{N}, p_n) summable to s .

If

$$(1.4) \quad (E, q) = E_n^q = \frac{1}{(1+q)^n} \sum_{\nu=0}^n \binom{n}{\nu} q^{n-\nu} s_\nu \rightarrow s, \text{ as } n \rightarrow \infty.$$

Then the series $\sum a_n$ is said to be summable (E, q) to a definite number s .

The product of (\bar{N}, p_n) summability with (E, q) summability defines $(\bar{N}, p_n)(E, q)$ summability and

we denote it by $\bar{N}_p E_n^q$ then if

$$(1.5) \quad \bar{N}_p E_n^q = \frac{1}{P_n} \sum_{k=0}^n \frac{p_k}{(1+q)^k} \sum_{\nu=0}^k \binom{k}{\nu} q^{k-\nu} s_\nu \rightarrow s, \text{ as } n \rightarrow \infty.$$

Then the series $\sum a_n$ is said to be summable to s by the $(\bar{N}, p_n)(E, q)$ method.

It is known that (E, q) is regular. It is supposed through out this paper that the method $(\bar{N}, p_n)(E, q)$ is regular.

Let $f(t)$ be a periodic function with period 2π , integrable in the sense of Lebesgue over $(-\pi, \pi)$, then

$$(1.6) \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

is the Fourier series associated with f .

We use the following notation through out this paper

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$K_n(t) = \frac{1}{2\pi P_n} \sum_{k=0}^n \frac{p_k}{(1+q)^k} \sum_{v=0}^k \binom{k}{v} q^{k-v} \frac{\sin\left(v + \frac{1}{2}\right)t}{\sin \frac{t}{2}}.$$

2. Known Theorem:

Dealing with $(N, p_n)(E, q)$ method of a Fourier series, Nigam, et.al[2] proved the following theorem:

Theorem-2.1:

Let $\{p_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty, \text{ as } n \rightarrow \infty.$$

If

$$(2.1.1) \quad \Phi(t) = \int_0^t |\phi(u)| du = O\left\{ \frac{t}{\alpha\left(\frac{1}{t}\right)} \right\}, \text{ as } t \rightarrow +0$$

and

$$(2.1.2) \quad \alpha(n) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

where $\alpha(t)$ be a positive, non-increasing function of t , then the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is summable $(N, p_n)(E, q)$ to $f(x)$ at the point $t = x$.

In this paper, we have generalized it to $(\bar{N}, p_n)(E, q)$ summability of Fourier series.

3. Main theorem:

Theorem -3.1:

Let $\{p_n\}$ a positive, monotonic, non-increasing sequence of real constants satisfying (1.1) and

$$(3.1.1) \quad \Phi(t) = \int_0^t |\phi(u)| du = O\left\{ \frac{t}{\alpha\left(\frac{1}{t}\right)} \right\}, \text{ as } t \rightarrow +0$$

and

$$(3.1.2) \quad \alpha(n) \rightarrow \infty \text{ as } n \rightarrow \infty$$

where $\alpha(t)$ is positive, non-increasing function of t , then the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is summable

$(\bar{N}, p_n)(E, q)$ at the point t .

4. Required Lemmas:

We require the following Lemmas to prove the theorem

Lemma -4.1:

$$|K_n(t)| = O(n), \quad 0 \leq t \leq \frac{1}{n+1}.$$

Proof:

For $0 \leq t \leq \frac{1}{n+1}$, we have $\sin nt \leq n \sin t$. Then

$$\begin{aligned} |K_n(t)| &= \frac{1}{2\pi P_n} \left| \sum_{k=0}^n \frac{p_k}{(1+q)^k} \sum_{v=0}^k \binom{k}{v} q^{k-v} \frac{\sin\left(v + \frac{1}{2}\right)t}{\sin \frac{t}{2}} \right| \\ &\leq \frac{1}{2\pi P_n} \left| \sum_{k=0}^n \frac{p_k}{(1+q)^k} \sum_{v=0}^k \binom{k}{v} q^{k-v} \frac{(2v+1)\sin \frac{t}{2}}{\sin \frac{t}{2}} \right| \\ &\leq \frac{1}{2\pi P_n} \left| \sum_{k=0}^n \frac{p_k(2k+1)}{(1+q)^k} \sum_{v=0}^k \binom{k}{v} q^{k-v} \right| \\ &= \frac{1}{2\pi P_n} \left| \sum_{k=0}^n \frac{p_k(2k+1)}{(1+q)^k} (1+q)^k \right|, \text{ as } \sum_{v=0}^k \binom{k}{v} q^{k-v} = (1+q)^k \\ &= \frac{(2n+1)}{2\pi P_n} \left| \sum_{k=0}^n p_k \right| \\ &= O(n) \end{aligned}$$

Lemma-4.2:

$$|K_n(t)| = O\left(\frac{1}{t}\right), \quad \text{for } \frac{1}{n} \leq t \leq \pi.$$

Proof:

For $\frac{1}{n} \leq t \leq \pi$, we have by Jordan's lemma, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$, $\sin nt \leq 1$.

Then

$$\begin{aligned} |K_n(t)| &\leq \frac{1}{2\pi P_n} \left| \sum_{k=0}^n \frac{p_k}{(1+q)^k} \sum_{v=0}^k \binom{k}{v} q^{k-v} \frac{\pi}{t} \right| \\ &= \frac{1}{2t P_n} \sum_{k=0}^n \frac{p_k}{(1+q)^k} (1+q)^k \\ &= \frac{1}{2t P_n} \sum_{k=0}^n p_k \\ &= O\left(\frac{1}{t}\right). \end{aligned}$$

5. Proof of the theorem- 3.1:

If $s_n(f; x)$ is the n-th partial sum of the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ of $f(t)$, then by using Riemann-Lebesgue theorem, following Titchmarsh [3] we have

$$s_n(f; x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} dt$$

Thus, the (E, q) transform E_n^q of $s_n(f; x)$ is given by

$$E_n^q - f(x) = \frac{1}{2\pi(1+q)^n} \int_0^{\pi} \frac{\phi(t)}{\sin\left(\frac{t}{2}\right)} \left\{ \sum_{k=0}^n \binom{n}{k} q^{n-k} \sin\left(k + \frac{1}{2}\right)t \right\} dt.$$

If $\bar{N}_p E_n^q$ denote the $(\bar{N}, p_n)(E, q)$ transform of $s_n(f; x)$, we then prove

$$\begin{aligned} \overline{N}_p E_n^q - f(x) &= \frac{1}{2\pi P_n} \sum_{k=0}^n \frac{P_k}{(1+q)^k} \int_0^\pi \frac{\phi(t)}{\sin\left(\frac{t}{2}\right)} \left\{ \sum_{\nu=0}^k \binom{k}{\nu} q^{k-\nu} \sin\left(\nu + \frac{1}{2}\right)t \right\} dt \\ &= \int_0^\pi \phi(t) K_n(t) dt \end{aligned}$$

In order to prove the theorem, under our assumption, it is sufficient to show that

$$\int_0^\pi \phi(t) K_n(t) dt = O(1) \quad \text{as } n \rightarrow \infty$$

For $0 < \delta < \pi$, we have $\overline{N}_p E_n^q - f(x) = \int_0^\pi \phi(t) K_n(t) dt$

$$\begin{aligned} &= \left(\int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi \right) \phi(t) K_n(t) dt \\ &= I_1 + I_2 + I_3 \quad , \text{ Say.} \end{aligned}$$

Now $|I_1| = \left| \int_0^{1/n} \phi(t) K_n(t) dt \right| \leq \int_0^{1/n} |\phi(t)| |K_n(t)| dt.$

$$\begin{aligned} &\leq O(n) \int_0^{1/n} |\phi(t)| dt \quad , \text{ using Lemma -1} \\ &= O(n) \left\{ O\left(\frac{1}{n\alpha(n)}\right) \right\} \quad , \text{ using (3.1.1).} \\ &= O\left(\frac{1}{\alpha(n)}\right) \quad , \quad \text{as } n \rightarrow \infty . \\ &= O(1) \quad , \quad \text{as } n \rightarrow \infty \quad , \text{ using (3.1.2).} \end{aligned}$$

Next

$$\begin{aligned} |I_2| &\leq \left| \int_{1/n}^\delta |\phi(t)| |k_n(t)| dt \right| \\ &= O \left\{ \int_{1/n}^\delta \frac{|\phi(t)|}{t} dt \right\} \quad , \text{ using lemma -2} \end{aligned}$$

$$= O\left\{\left[\frac{\Phi(t)}{t}\right]_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{\Phi(t)}{t^2} dt\right\}.$$

$$= O\left\{o\left[\frac{1}{\alpha\left(\frac{1}{t}\right)}\right]_{1/n}^{\delta} + \int_{1/\delta}^n o\left(\frac{1}{u\alpha(u)}\right) du\right\}, \text{ where } u=1/t$$

and $0 < \delta < 1$.

$$= o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{n\alpha(n)}\right) \int_{1/\delta}^n du, \text{ using second mean-value}$$

theorem for the integral in the 2nd term as $\alpha(n)$ is monotonic.

$$= O(1) + O(1) \text{ as } n \rightarrow \infty, \text{ using (3.1.2)}$$

$$= O(1), \text{ as } n \rightarrow \infty.$$

Finally

$$|I_3| \leq \int_{\delta}^{\pi} |\phi(t)| |K_n(t)| dt = O(1) \text{ as } n \rightarrow \infty.$$

by using Riemann –Lebesgue theorem and the regularity condition of the method of summability.

Then, $\bar{N}_p E_n^q - f(x) = O(1)$, as $n \rightarrow \infty$.

This completes the proof of the theorem.

6. Corollaries:

The following corollaries can be derived from our main theorem,

Corollary-6.1: If $\Phi(t) = \int_0^t |\phi(u)| du = O\left\{\frac{t}{\alpha\left(\frac{1}{t}\right)}\right\}$, as $t \rightarrow +0$ and $\alpha(n) \rightarrow \infty$ as $n \rightarrow \infty$ where

$\alpha(t)$ be a positive, non-increasing function of t , then the series $\sum_{n=0}^{\infty} A_n(x)$ is summable

(C,1)(E,q) to $f(x)$ at the point $t = x$.

Proof:

If we take $p_n = 1, \forall n$, the result follows from the main theorem.

Corollary - 6.2:

If $\Phi(t) \equiv 0 \left\{ \frac{t}{\log \left(\frac{1}{t} \right)} \right\}$, as $t \rightarrow +0$ then the Fourier series (1.6) is $(\overline{N}, p_n)(E, q)$

summable to $f(x)$.

Under the hypothesis, condition (3.1.2) is obviously satisfied on $\log n \rightarrow \infty$ as $n \rightarrow \infty$.

Corollary - 6.3:

If $\Phi(t) \equiv 0(t)$, as $t \rightarrow +0$, then the Fourier series $(\overline{N}, p_n)(E, q)$ summable to $f(x)$ without employing (3.1.2).

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